

Directed transport of Brownian particles in a changing temperature field

This article has been downloaded from IOPscience. Please scroll down to see the full text article.

2008 J. Phys. A: Math. Theor. 41 015002

(<http://iopscience.iop.org/1751-8121/41/1/015002>)

View [the table of contents for this issue](#), or go to the [journal homepage](#) for more

Download details:

IP Address: 171.66.16.147

The article was downloaded on 03/06/2010 at 06:34

Please note that [terms and conditions apply](#).

Directed transport of Brownian particles in a changing temperature field

A Grillo^{1,2}, A Jinha³, S Federico³, R Ait-Haddou³, W Herzog³
and G Giaquinta^{1,2}

¹ DMFCI, Facoltà di Ingegneria, Università di Catania. Viale Andrea Doria 6, 95125 Catania, Italy

² CNISM, Unità di Ricerca di Catania. Viale Andrea Doria 6, 95125 Catania, Italy

³ HPL—Faculty of Kinesiology, The University of Calgary, 2500 University Drive NW, Calgary, Alberta, T2N 1N4, Canada

E-mail: agrillo@dmfci.unict.it

Received 19 June 2007, in final form 9 October 2007

Published 12 December 2007

Online at stacks.iop.org/JPhysA/41/015002

Abstract

We study the interaction of Brownian particles with a changing temperature field in the presence of a one-dimensional periodic adiabatic potential. We show the existence of directed transport through the determination of the overall current of Brownian particles crossing the boundary of the system. With respect to the case of Brownian particles in a thermal bath, we determine a current which exhibits a contribution explicitly related to the presence of a thermal gradient. Beyond the self-consistent calculation of the temperature and probability density distribution of Brownian particles, we evaluate the energy consumption for directed transport to take place. Our description is based on Streater's model, and solutions are obtained by perturbing the system from its initial thermodynamic equilibrium state.

PACS numbers: 05.60.-k, 05.40.Jc

(Some figures in this article are in colour only in the electronic version)

1. Introduction

In recent years, there has been an increasing tendency toward the use of *Brownian motors*, or *ratchets*, to describe a variety of physical systems either of biological or technological relevance. In biophysics, Brownian motors are a suitable 'candidate' to explain phenomena such as muscle contraction, molecular transport and some functions of biological ion channels [1–3]. In the technological context, ratchets are used to describe phenomena such as mass separation, trapping schemes and directed vortex motion in superconductors [4–6].

The concept on which the theory of Brownian motors has been developed lies in the possibility of gaining useful work from *unbiased* microscopic random fluctuations. This idea was originally put forward by Smoluchowski and Feynman [7], and the explanation and physical interpretation of the Smoluchowski and Feynman's experiment can be found, for example, in [8].

The ability of extracting useful work from a Brownian system is related to the notion of directed transport of Brownian particles. Directed transport consists of a dissipative dynamics in the presence of thermal noise and some perturbation that drives the system away from equilibrium without introducing an *a priori* biased direction of motion [8]. These perturbations can be deterministic or stochastic, and can be externally imposed or intrinsic to the system. Brownian motors are studied in spatially periodic small-scale systems. In this case, directed transport is achieved by *breaking* thermal equilibrium and *spatial inversion symmetry* [8, 9]. As pointed out by Reimann [8], symmetry breaking may occur in three ways: (i) intrinsically, i.e. in the presence of a periodic asymmetric potential (the so-called ratchet potential); (ii) by means of non-equilibrium perturbations which bring about a spatial asymmetry of the dynamics or (iii) through spontaneous symmetry breaking in the form of collective effects in coupled, perfectly symmetric, non-equilibrium systems.

Several types of Brownian motors have been identified. *Rocket*, or *thermal*, ratchets consist of Brownian particles able to rectify noise into vectorial motion in an anisotropic potential undergoing external fluctuations or external periodic forces [10, 11]. In *flashing* ratchets, Brownian particles are subject to switching potentials, and the direction of motion depends on the rate of switching and the asymmetry of the potentials [11, 12]. Lastly, in *colored* ratchets, transport of Brownian particles is produced by inducing transport from the time correlation of colored noise in an anisotropic potential [10].

An important feature of Brownian motors was outlined by Magnasco [13] who showed that Brownian particles can exhibit a non-zero net drift speed when subjected to a symmetry breaking potential and a time-correlated external force. Parrondo [14] proved the existence of transport of Brownian particles in a one-dimensional periodic potential which changes adiabatically in time. By letting the potential depend on time through a certain collection of parameters, Parrondo was able to express the net fraction of particles crossing the boundary of the system (during an adiabatic variation of the potential) in terms of the contour integral of a non-exact differential form in the space of parameters of the potential. The work required to change the potential is an exact differential form in the space of parameters, and a closed path in this space produces transport of particles without energy consumption. These cycles were therefore named *reversible ratchets* [14].

The results presented in [14] were obtained by studying the Smoluchowski equation,

$$\frac{\partial \rho}{\partial t} = \frac{\partial}{\partial x} \left(\frac{k_B T}{\zeta} \frac{\partial \rho}{\partial x} + \frac{1}{\zeta} \frac{\partial V}{\partial x} \rho \right), \quad (1.1)$$

where ρ denoted the probability density distribution of Brownian particles, k_B the Boltzmann constant, T absolute temperature, ζ a damping constant and V the adiabatic ratchet potential. Equation (1.1) was investigated in a one-dimensional periodic lattice and assuming that the potential, V , was periodic in time and space, and that its period in space coincided with the size of the lattice reference cell. This allowed for investigating equation (1.1) in the domain $[x_n, x_n + L]$, where x_n and L denoted the n th lattice site and the size of the representative cell, respectively. Consequently, the ratchet potential was introduced by means of the function $V : \mathbb{R}_0^+ \times [x_n, x_n + L] \rightarrow \mathbb{R}$, defined at every time $t \in \mathbb{R}_0^+$ and space coordinate $x \in [x_n, x_n + L]$. The assumption of an adiabatic potential enabled Parrondo to find the solution to equation (1.1) as the sum of an equilibrium probability density distribution (given by the Gibbs probability

density distribution) and a field φ whose integral over the interval $[x_n, x_n + L]$ was imposed to be zero. Both the Gibbs probability density distribution and the field φ were found by enforcing periodic boundary conditions on the boundary of the lattice reference cell.

The Smoluchowski equation (1.1) represents the Fokker–Planck equation corresponding to the Langevin description of motion for a Brownian particle, i.e.

$$\zeta \dot{z}(t) = -\frac{\partial V}{\partial x}(t, z(t)) + \xi(t). \quad (1.2)$$

Here, $z(t)$ denotes the trajectory of the Brownian particle and $\xi(t)$ is the zero-mean Gaussian white thermal noise (i.e. $\langle \xi(t) \rangle = 0$) whose time correlation function is given by the dissipation–fluctuation relation $\langle \xi(t) \xi(t') \rangle = 2k_B T \delta(t - t')$. Equation (1.2) holds true when the Brownian particle is assumed to *thermalize* rapidly. Since the characteristic time scale for thermalization is defined by $t_{\text{th}} = m/\zeta$ (m is the mass of a Brownian particle), when $t \gg t_{\text{th}}$, the inertial term, $m\ddot{z}(t)$, becomes irrelevant and the friction force, $\zeta \dot{z}(t)$, is compensated for by the external random forces. In this case, the velocity $\dot{z}(t)$ is said to be the *terminal velocity*. In this framework, temperature is regarded as a constant (this condition can be achieved either by a controlled heat flow across the boundary of the system or under the hypothesis of infinite thermal conductivity of the medium in which the Brownian particles evolve) and leads to the constant molecular diffusivity $D = (k_B T)/\zeta$ featuring in equation (1.1).

In this paper, we propose an extension of Parrondo’s results [14], by assuming that temperature changes in both time and space. In this case, if the system of interest is closed (in the sense that it does not exchange heat with the outside world), the motion of a particle leaves a trail of heat which, being distributed to the whole system, produces a local change in the temperature of the system. On the other hand, a variation in the temperature field influences the behavior of a Brownian particle by changing its diffusivity and probability density distribution. Therefore, the equation of motion of the particle must be supplemented and coupled with a heat-dependent equation of motion. Streater [15, 16] suggested a model of a Brownian particle in a closed system in which changes in temperature are considered and the first and second law of thermodynamics are satisfied. On the basis of Streater’s model, it is possible to study the interaction of Brownian particles with a changing temperature field in the presence of a one-dimensional periodic adiabatic potential. Under these hypotheses, we prove the existence of directed transport and show how the variability of temperature plays a role in the determination of the net current of particles crossing the boundary of the system. This current will be evaluated after the self-consistent calculation of the temperature and probability distribution of Brownian particles. By virtue of the adiabaticity of the potential, the solutions of the coupled partial differential equations given by Streater’s model will be found by perturbing the system from its thermodynamic equilibrium state and will be expressed in terms of asymptotic expansions.

For our purposes, the time scales characterizing thermal diffusion, t_T , diffusion of Brownian particles, t_D , and the periodic variation of the potential, t_V , will be introduced and compared with each other. We remark that, since our long-term goal is the development of a model capable of describing molecular motors in biological applications, and we are particularly interested in specifying our model to the study of muscle contraction, we shall assume that the time scale t_D refers to the diffusion of relatively big molecules, and that, through a suitable choice of the parameters defining t_T , the time scale t_T is smaller than t_D .

Before going further, we would like to remark that there are rather recent papers (cf, for example [17, 18]) in which the existence of unidirectional motion for a classical Brownian particle subject to the action of a one-dimensional parametric potential is proven. In these papers, the potential, V , is defined as function of space, and is parameterized by a time-varying quantity, $\gamma(t)$, such that $V(x, \gamma(t)) = W(x) + W(x - \gamma(t))$. The process $\gamma(t)$, named *external*

translations by these authors, admits an effective counterpart, denoted by $\Gamma(t)$ in [18], that satisfies a Langevin-type equation, and whose values identify the irregularities encountered by the trajectory of the Brownian particle. The equation governing the evolution of the Brownian particle is thus coupled with the equation defining the behavior of $\Gamma(t)$ through the definition of the potential, $V(x, \gamma(t))$.

2. Streater's model

According to Streater's model [15, 16], the Smoluchowski equation (1.1) is coupled with a *nonlinear* Fourier equation for the temperature field, T , i.e.

$$\begin{cases} \frac{\partial \rho}{\partial t} = \frac{\partial}{\partial x} \left[\frac{k_B T}{\zeta} \frac{\partial \rho}{\partial x} + \frac{1}{\zeta} \frac{\partial V}{\partial x} \rho \right] \\ C \frac{\partial T}{\partial t} = \lambda \frac{\partial^2 T}{\partial x^2} + \frac{\partial V}{\partial x} \left[\frac{k_B T}{\zeta} \frac{\partial \rho}{\partial x} + \frac{1}{\zeta} \frac{\partial V}{\partial x} \rho \right] \end{cases} \quad (2.1)$$

where specific heat, C , and thermal conductivity, λ , are assumed to be constant, and molecular diffusivity is assumed to depend on the temperature field through the linear relation $D(T(t, x)) = (k_B T(t, x))/\zeta$. We note that the nonlinearity of equations (2.1) is because the coupling term, given by the probability current

$$J = - \left[\frac{k_B T}{\zeta} \frac{\partial \rho}{\partial x} + \frac{1}{\zeta} \frac{\partial V}{\partial x} \rho \right], \quad (2.2)$$

depends on temperature in a nonlinear way.

We shall investigate the interaction between Brownian particles (characterized by the probability density distribution ρ) and the changing temperature field, T , in one-dimensional periodic lattice. Then, by letting x_n and L denote the n th lattice site and the size of the lattice reference cell, respectively, the domain in which we shall study our problem is the reference cell $[x_n, x_n + L]$. Similar to Parrondo's approach [14], we shall consider the adiabatic ratchet potential, V , to be periodic in time and space. In particular, the space period will be assumed to coincide with the length of the reference cell, L .

For our purposes, we supplement equations (2.1) with the following initial and boundary conditions:

Initial and boundary conditions imposed on the probability density distribution, ρ :

$$\rho(t_{\text{in}}, x) = \rho_{\text{in}}(x), \quad \forall x \in [x_n, x_n + L], \quad (2.3)$$

$$\rho(t, x_n) = \rho(t, x_n + L), \quad \forall t \in \mathbb{R}_0^+, \quad (2.4)$$

$$\int_{x_n}^{x_n+L} \rho(t, x) dx = 1, \quad \forall t \in \mathbb{R}_0^+. \quad (2.5)$$

Initial and boundary conditions imposed on the temperature field, T :

$$T(t_{\text{in}}, x) = T^{\text{in}}, \quad \forall x \in [x_n, x_n + L], \quad (2.6)$$

$$\begin{cases} T(t, x_n) = T(t, x_n + L), & \forall t \in \mathbb{R}_0^+, \\ \frac{\partial T}{\partial x}(t, x_n) = \frac{\partial T}{\partial x}(t, x_n + L), & \forall t \in \mathbb{R}_0^+. \end{cases} \quad (2.7)$$

The boundary conditions (2.4) and (2.7) require the probability density distribution, temperature and temperature gradient to be periodic functions of space, while equation (2.5)

imposes the normalization condition on the probability density distribution. We remark that, although in equation (2.6) we imposed that the initial temperature is equal to the constant T^{in} , the results presented in this paper hold true even in the more general case in which the initial temperature distribution equals a function of space coordinates, i.e. $T(t_{\text{in}}, x) = T_{\text{in}}(x)$.

In order to verify that the first law of thermodynamics is satisfied, we need to evaluate the derivative of energy with respect to time. Since the global energy of the system is given by

$$E(t) = \int_{x_n}^{x_n+L} \rho(t, x) V(t, x) dx + C \int_{x_n}^{x_n+L} T(t, x) dx, \quad (2.8)$$

differentiation with respect to time, integration by parts, and taking into account equations (2.1) yield

$$\frac{dE}{dt}(t) = -[J(t, x)V(t, x)]_{x_n}^{x_n+L} + \left[\lambda \frac{\partial T}{\partial x}(t, x) \right]_{x_n}^{x_n+L} + \int_{x_n}^{x_n+L} \rho(t, x) \frac{\partial V}{\partial t}(t, x) dx. \quad (2.9)$$

If the potential V is periodic, a further simplification occurs if the probability current, J , and the temperature gradient, $\partial T/\partial x$, are spatially periodic functions in the interval $[x_n, x_n + L]$. In this case, equation (2.9) becomes

$$\frac{dE}{dt}(t) = \int_{x_n}^{x_n+L} \rho(t, x) \frac{\partial V}{\partial t}(t, x) dx. \quad (2.10)$$

Equation (2.10) illustrates that the energy of the system is not conserved. This means that, for the system under investigation, the first law of thermodynamics does not reduce to the strict balance of energy, i.e.

$$\frac{dE}{dt}(t) = 0. \quad (2.11)$$

In order to check the second law of thermodynamics, we need to write an expression for entropy, i.e.

$$S(t) = -k_B \int_{x_n}^{x_n+L} \rho(t, x) \ln[\rho(t, x)] dx + C \int_{x_n}^{x_n+L} \ln \left(\frac{T(t, x)}{\vartheta} \right) dx, \quad (2.12)$$

where ϑ is a constant reference temperature. By differentiating with respect to time, using equations (2.1) and assuming that the potential, V , the probability current, J , and the temperature gradient, $\partial T/\partial x$, are spatially periodic functions in the interval $[x_n, x_n + L]$, we find

$$\frac{dS}{dt}(t) = k_B \int_{x_n}^{x_n+L} \frac{[J(t, x)]^2}{D(T(t, x))\rho(t, x)} dx + \lambda \int_{x_n}^{x_n+L} \left[\frac{1}{T(t, x)} \frac{\partial T}{\partial x}(t, x) \right]^2 dx \geq 0. \quad (2.13)$$

Since both terms on the right-hand side of equation (2.13) are greater than or equal to zero, the second law of thermodynamics is always respected.

3. Perturbative approach to Streater's model

In this section, we search for the solution to equations (2.1) by perturbing the system from its thermodynamic equilibrium state. Our approach is based on the possibility of determining two time scales, t_f and t_s , which refer to the *fastest* and the *slowest* process occurring in the system, respectively. We hypothesize that these time scales are well separated, i.e. we postulate the existence of a positive real smallness parameter, ε , such that $\varepsilon = t_f/t_s$.

The ratchet potential, V , is periodic in time with period t_V , and we require that its *adiabatic* variation over t_V represents the slowest process accounted for in the system. This statement

implies that $t_s = t_V$. Following the picture presented by Parrondo [14], by *slow* (or *adiabatic*) variation of the potential over its period, we mean that, if the interval over which the potential varies in time is discretized into N steps and the potential is assumed to make a ‘jump’ at each step, then, after each jump, we allow the system to relax before the potential makes the following jump. According to this description, the relaxation time has to be identified with the characteristic time scale associated with the fastest process taking place in the system, i.e. t_f , and the adiabatic limit is attained by imposing $t_f/t_V = \varepsilon$, and letting the potential change smoothly in time.

In the case studied by Parrondo [14], temperature was regarded as a constant, and the only equation to be solved was the Smoluchowski equation (cf equation (1.1)). Therefore, the relaxation time of the system coincided with the time scale associated with diffusion of Brownian particles. In our case, instead, it is necessary to distinguish between the relaxation time associated with thermal diffusion (i.e., the time scale associated with thermal diffusion) and the relaxation time referring to the diffusion of Brownian particles (i.e., the time scale associated with molecular diffusivity). Since, as anticipated in section 1, we assume in this paper that the former (denoted by t_T) is smaller than the latter (denoted by t_D), we regard thermal diffusion as the fastest process accounted for in the system under investigation. Thus, we set $t_f = t_T$, and we base our discussion on the condition $t_T/t_V = \varepsilon \ll 1$.

Since we are interested in investigating the behavior of the system at long time scales (i.e. at the time scale dictated by t_V), we shall rescale the system of PDEs given in equation (2.1), together with the initial and boundary conditions stated in equations (2.3)–(2.7), and study it at the time scale at which the temporal variation of the adiabatic ratchet potential becomes $O(1)$.

For our purposes, we introduce the nondimensional time variable $\tau = t/t_V$ that is meant to account for the behavior of the system in response to the slow process characterized by the time scale t_V . Moreover, we decompose each physical quantity featuring in equations (2.1) as the product of a *characteristic* part (denoted by the index c) and an *intrinsic* part (denoted by an overbar), i.e.

$$\rho(t, x) = \rho_c \bar{\rho}(\tau, X), \quad T(t, x) = T_c \bar{T}(\tau, X) \quad \text{and} \quad V(t, x) = V_c \bar{V}(\tau, X), \quad (3.1)$$

where $X = x/L \in [X_n, X_{n+1}]$ is the nondimensional space coordinate (in nondimensional units, the lattice constant is equal to unity). By rewriting equations (2.1) with the formalism presented in equations (3.1) and noticing that the operator $\partial/\partial t$ rescales as

$$\frac{\partial}{\partial t} = \frac{1}{t_V} \frac{\partial}{\partial \tau}, \quad (3.2)$$

we obtain

$$\begin{cases} \frac{1}{t_V} \frac{\partial \bar{\rho}}{\partial \tau} = \frac{k_B T_c}{\zeta L^2} \frac{\partial}{\partial X} \left[\bar{T} \frac{\partial \bar{\rho}}{\partial X} + \frac{V_c}{k_B T_c} \frac{\partial \bar{V}}{\partial X} \bar{\rho} \right] \\ \frac{1}{t_V} \frac{\partial \bar{T}}{\partial \tau} = \frac{\lambda}{CL^2} \frac{\partial^2 \bar{T}}{\partial X^2} + \frac{\rho_c V_c k_B T_c}{CT_c \zeta L^2} \frac{\partial \bar{V}}{\partial X} \left[\bar{T} \frac{\partial \bar{\rho}}{\partial X} + \frac{V_c}{k_B T_c} \frac{\partial \bar{V}}{\partial X} \bar{\rho} \right]. \end{cases} \quad (3.3)$$

By recognizing that the ratios $(k_B T_c)/(\zeta L^2)$ and $\lambda/(CL^2)$ represent the inverse of the characteristic time scales associated with molecular diffusivity of Brownian particles, t_D , and thermal diffusion, t_T , respectively (i.e. $t_D = (\zeta L^2)/(k_B T_c)$ and $t_T = (CL^2)/\lambda$), we can rewrite equations (3.3) as

$$\begin{cases} \frac{1}{t_V} \frac{\partial \bar{\rho}}{\partial \tau} = \frac{1}{t_D} \frac{\partial}{\partial X} \left[\bar{T} \frac{\partial \bar{\rho}}{\partial X} + \phi_c \frac{\partial \bar{V}}{\partial X} \bar{\rho} \right] \\ \frac{1}{t_V} \frac{\partial \bar{T}}{\partial \tau} = \frac{1}{t_T} \frac{\partial^2 \bar{T}}{\partial X^2} + \frac{1}{t_D} \mu \frac{\partial \bar{V}}{\partial X} \left[\bar{T} \frac{\partial \bar{\rho}}{\partial X} + \phi_c \frac{\partial \bar{V}}{\partial X} \bar{\rho} \right], \end{cases} \quad (3.4)$$

where $\phi_c = V_c/(k_B T_c)$ and $\mu = (k_B \rho_c \phi_c)/C$. Since we assume that the characteristic time scale of thermal diffusion, t_T , is smaller than the time scale associated with molecular diffusivity, t_D , by introducing the ratio $\sigma = t_T/t_D < 1$ and the positive smallness parameter $\varepsilon = t_T/t_V$, we can rearrange equations (3.4) as

$$\begin{cases} \frac{\partial \bar{\rho}}{\partial \tau} = \frac{1}{\varepsilon} \sigma \frac{\partial}{\partial X} \left[\bar{T} \frac{\partial \bar{\rho}}{\partial X} + \phi_c \frac{\partial \bar{V}}{\partial X} \bar{\rho} \right] \\ \frac{\partial \bar{T}}{\partial \tau} = \frac{1}{\varepsilon} \frac{\partial^2 \bar{T}}{\partial X^2} + \frac{1}{\varepsilon} \mu \sigma \frac{\partial \bar{V}}{\partial X} \left[\bar{T} \frac{\partial \bar{\rho}}{\partial X} + \phi_c \frac{\partial \bar{V}}{\partial X} \bar{\rho} \right]. \end{cases} \quad (3.5)$$

Since we are interested in performing an asymptotic analysis of equations (3.5), we expand the probability density distribution, $\bar{\rho}$, and temperature, \bar{T} , in asymptotic series, i.e.

$$\bar{\rho}(\tau, X) = \sum_{j=0}^{\infty} \varepsilon^j \bar{\rho}_j(\tau, X) \quad \text{and} \quad \bar{T}(\tau, X) = \sum_{j=0}^{\infty} \varepsilon^j \bar{T}_j(\tau, X). \quad (3.6)$$

We assume that expansions (3.6) exist and converge to their asymptotic solution in the limit $\varepsilon \ll 1$. If these expressions are limited to the first order, the current of Brownian particles is approximated by

$$J = \frac{\rho_c k_B T_c}{L \zeta} [\bar{J}_0 + \varepsilon \bar{J}_1 + \varepsilon^2 \bar{J}_2 + \dots], \quad (3.7)$$

where the quantities

$$\begin{aligned} \bar{J}_0 &= - \left[\bar{T}_0 \frac{\partial \bar{\rho}_0}{\partial X} + \phi_c \frac{\partial \bar{V}}{\partial X} \bar{\rho}_0 \right], \\ \bar{J}_1 &= - \left[\bar{T}_0 \frac{\partial \bar{\rho}_1}{\partial X} + \bar{T}_1 \frac{\partial \bar{\rho}_0}{\partial X} + \phi_c \frac{\partial \bar{V}}{\partial X} \bar{\rho}_1 \right], \\ \bar{J}_2 &= - \left[\bar{T}_1 \frac{\partial \bar{\rho}_1}{\partial X} \right] \end{aligned} \quad (3.8)$$

represent the nondimensional zeroth-, first- and second-order terms of the asymptotic expansion of the probability current, i.e.

$$\bar{J} = \sum_{j=0}^{\infty} \varepsilon^j \bar{J}_j. \quad (3.9)$$

By accounting for equations (3.8), substituting the asymptotic expansions (3.6) into equation (3.5) and collecting terms of the same power of ε , we obtain three sets of equations that have to be satisfied asymptotically for any infinitesimally small ε . Since the coefficients of each power of ε are independent of the parameter itself, each coefficient must be identically zero in order to satisfy the equation for any arbitrarily small ε . Thus, by requiring the coefficients of the orders $O(\varepsilon^{-1})$, $O(1)$ and $O(\varepsilon)$ to be identically zero, we can reformulate Streater's model in terms of three sets of nondimensional equations, i.e.

$$\varepsilon^{-1} : \begin{cases} 0 = \sigma \frac{\partial}{\partial X} \left[\bar{T}_0 \frac{\partial \bar{\rho}_0}{\partial X} + \phi_c \frac{\partial \bar{V}}{\partial X} \bar{\rho}_0 \right], \\ 0 = \frac{\partial^2 \bar{T}_0}{\partial X^2} + \sigma \mu \frac{\partial \bar{V}}{\partial X} \left[\bar{T}_0 \frac{\partial \bar{\rho}_0}{\partial X} + \phi_c \frac{\partial \bar{V}}{\partial X} \bar{\rho}_0 \right], \end{cases} \quad (3.10)$$

$$\varepsilon^0 : \begin{cases} \frac{\partial \bar{\rho}_0}{\partial \tau} = \sigma \frac{\partial}{\partial X} \left[\bar{T}_0 \frac{\partial \bar{\rho}_1}{\partial X} + \bar{T}_1 \frac{\partial \bar{\rho}_0}{\partial X} + \phi_c \frac{\partial \bar{V}}{\partial X} \bar{\rho}_1 \right], \\ \frac{\partial \bar{T}_0}{\partial \tau} = \frac{\partial^2 \bar{T}_1}{\partial X^2} + \sigma \mu \frac{\partial \bar{V}}{\partial X} \left[\bar{T}_0 \frac{\partial \bar{\rho}_1}{\partial X} + \bar{T}_1 \frac{\partial \bar{\rho}_0}{\partial X} + \phi_c \frac{\partial \bar{V}}{\partial X} \bar{\rho}_1 \right], \end{cases} \quad (3.11)$$

$$\varepsilon : \begin{cases} \frac{\partial \bar{\rho}_1}{\partial \tau} = \sigma \frac{\partial}{\partial X} \left[\bar{T}_1 \frac{\partial \bar{\rho}_1}{\partial X} \right], \\ \frac{\partial \bar{T}_1}{\partial \tau} = \frac{\partial^2 \bar{T}_2}{\partial X^2} + \sigma \mu \frac{\partial \bar{V}}{\partial X} \left[\bar{T}_1 \frac{\partial \bar{\rho}_1}{\partial X} \right]. \end{cases} \quad (3.12)$$

Moreover, in order for the boundary conditions (2.4), and (2.7) to be satisfied, at any order, by all terms of the asymptotic expansions (3.6), we require that

$$\begin{aligned} \bar{\rho}_0(t, X_n) &= \bar{\rho}_0(t, X_n + 1), & \bar{\rho}_1(t, X_n) &= \bar{\rho}_1(t, X_n + 1), \\ \int_0^1 \bar{\rho}_0(t, X) dX &= 1, & \int_0^1 \bar{\rho}_1(t, X) dX &= 0, \end{aligned} \quad (3.13)$$

and

$$\begin{aligned} \bar{T}_0(t, X_n) &= \bar{T}_0(t, X_n + 1), & \bar{T}_1(t, X_n) &= \bar{T}_1(t, X_n + 1), \\ \bar{T}_2(t, X_n) &= \bar{T}_2(t, X_n + 1), & \frac{\partial \bar{T}_0}{\partial X}(t, X_n) &= \frac{\partial \bar{T}_0}{\partial X}(t, X_n + 1), \\ \frac{\partial \bar{T}_1}{\partial X}(t, X_n) &= \frac{\partial \bar{T}_1}{\partial X}(t, X_n + 1), & \frac{\partial \bar{T}_1}{\partial X}(t, X_n) &= \frac{\partial \bar{T}_1}{\partial X}(t, X_n + 1). \end{aligned} \quad (3.14)$$

We notice that, by rescaling time as $t \rightarrow t/t_T$, it is also possible to introduce the ‘rapidly’ varying time coordinate $\theta = t/t_T$. Such nondimensional variable describes the response of the system to processes characterized by the time scale t_T . Since, in general, the terms $\bar{\rho}_1$ and \bar{T}_1 of the asymptotic expansions (3.6) depend also on θ , equations (3.10)–(3.12) should be understood as a result of an average of equations (2.1) over the ‘rapidly’ varying time coordinate θ .

4. Solution of the expanded Streater’s equations

In order to determine the asymptotic solutions to equations (3.10)–(3.12), we need to specify the ratios ϕ_c , μ and σ , and the adiabatic potential \bar{V} . In this paper, we set $\phi_c = 1$ and $\mu = 1$, we let σ range between the smallness parameter, ε , and unity, and we choose the potential

$$\begin{aligned} \bar{V}(\tau, X) &= \{b + R_1(\tau) \cos(2\pi X)\} \{c + R_2(\tau) \cos[2\pi(X - a)]\} \\ &\quad - \{b + R_1(\tau)\} \{c + R_2(\tau) \cos(2\pi a)\}. \end{aligned} \quad (4.1)$$

In equation (4.1), we introduced the functions R_1 and R_2 to express the dependence of the potential on time and we used the constant a to make the potential *asymmetric*.

By assuming that the constants a , b , and c are fixed, the instantaneous shape of the potential as a function of space is determined by $R_1(\tau)$ and $R_2(\tau)$. Therefore, the functions $R_1(\tau)$ and $R_2(\tau)$ can be treated as parameters which, by varying adiabatically and periodically in time, provide the instantaneous spatial distribution of the potential. In this paper, we define $R_1(\tau)$ and $R_2(\tau)$ by the relations

$$R_1(\tau) = r_0 + r \cos(2\pi \tau), \quad R_2(\tau) = r_0 + r \sin(2\pi \tau). \quad (4.2)$$

In order for the potential (4.1) to be completely determined, the values of the constants a , b , c , r_0 and r must be assigned. The values chosen for these constants are listed in table 1, and the corresponding shape of the potential is shown in figure 1.

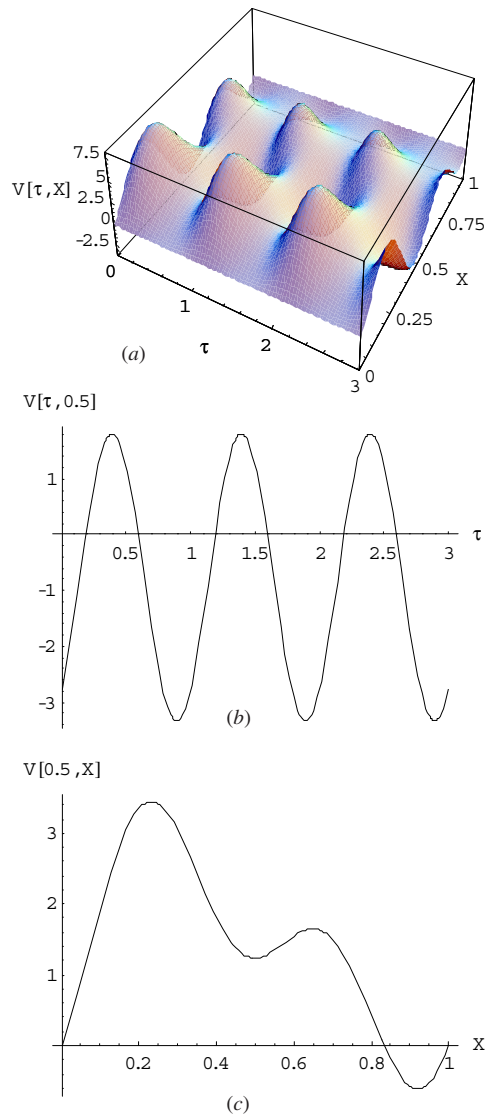


Figure 1. Shape of the potential \bar{V} given in equation (4.1). (a) The potential, V , is plotted in nondimensional units against time $\tau = t/t_V$ over three cycles (i.e. $\tau \in [0, 3]$) and space $X \in [X_n, X_n + 1]$. (b) The potential \bar{V} is evaluated at $X = 0.5$ and plotted against time. (c) The potential \bar{V} is evaluated at time $t = 0.5t_V$ (i.e. $\tau = 0.5$) and plotted against space.

Table 1. Constants which determine the potential given in equation (4.1)

a	0.4
b	1
c	1
r_0	2
r	1

4.1. Determination of the zeroth-order fields

The formal solution of equations (3.10) is given by

$$\bar{\rho}_0(\tau, X) = C_1(\tau)\gamma_-(\tau, X) - C_0(\tau)\gamma_-(\tau, X) \int_{X_n}^X \frac{\gamma_+(\tau, X')}{\bar{T}_0(\tau, X')} dX', \quad (4.3)$$

and

$$\bar{T}_0(\tau, X) = \sigma\mu C_0(\tau) \int_{X_n}^X \bar{V}(\tau, X') dX' + b_0(\tau)X + b_1(\tau). \quad (4.4)$$

The functions $\gamma_{\mp}(\tau, X)$ and the constants $C_0(\tau)$, $C_1(\tau)$ and $b_0(\tau)$ are defined by

$$\gamma_{\mp}(\tau, X) = \exp \left[\mp \phi_c \int_{X_n}^X \frac{1}{\bar{T}_0(\tau, X')} \frac{\partial \bar{V}}{\partial X}(\tau, X') dX' \right], \quad (4.5)$$

$$C_0(\tau) = \frac{\gamma_-(\tau, X_n + 1) - 1}{\Delta(\tau)}, \quad C_1(\tau) = \frac{\gamma_-(\tau, X_n + 1) \int_{X_n}^{X_n+1} \frac{\gamma_+(\tau, X')}{\bar{T}_0(\tau, X')} dX'}{\Delta(\tau)}, \quad (4.6)$$

$$b_0(\tau) = -\sigma\mu C_0(\tau) \int_{X_n}^{X_n+1} \bar{V}(\tau, X') dX', \quad (4.7)$$

where the quantity $\Delta(\tau)$ is equal to

$$\begin{aligned} \Delta(\tau) = & \left[\gamma_-(\tau, X_n + 1) \int_{X_n}^{X_n+1} \frac{\gamma_-(\tau, X')}{\bar{T}_0(\tau, X')} dX' \right] \int_{X_n}^{X_n+1} \gamma_-(\tau, X') dX' \\ & - [\gamma_-(\tau, X_n + 1) - 1] \left[\int_{X_n}^{X_n+1} \gamma_-(\tau, X_n + 1) \left(\int_{X_n}^{X'} \frac{\gamma_-(\tau, X'')}{\bar{T}_0(\tau, X'')} dX'' \right) dX' \right]. \end{aligned} \quad (4.8)$$

We notice that the constant $b_1(\tau)$ is still undetermined. It is also worth to remark that the constant $C_0(\tau)$ represents the zeroth-order current of Brownian particles, i.e. $C_0(\tau) = \bar{J}_0(\tau)$.

According to equations (4.3)–(4.8), the zeroth-order probability density distribution can be determined after temperature \bar{T}_0 is calculated. In order to do that, we write temperature \bar{T}_0 as the sum of its *mean value*

$$\bar{T}_0^m(\tau) = \int_{X_n}^{X_n+1} \bar{T}_0(\tau, X) dX, \quad (4.9)$$

and the corresponding fluctuation $\bar{T}_0^f(\tau, X) = \bar{T}_0(\tau, X) - \bar{T}_0^m(\tau)$.

The mean value \bar{T}_0^m can be computed by having recourse to the second equation of the set (3.11). Indeed, integrating both sides of this equation over the reference cell $[X_n, X_n + 1]$ and enforcing periodic boundary conditions on the first-order temperature gradient, $\partial \bar{T}_1 / \partial X$, lead to the self-consistency condition

$$\frac{d\bar{T}_0^m}{d\tau}(\tau) = -\mu \int_{X_n}^{X_n+1} \bar{V}(\tau, X) \frac{\partial \bar{\rho}_0}{\partial \tau}(\tau, X) dX. \quad (4.10)$$

Furthermore, by applying definition (4.9) to equation (4.4) and using equation (4.7), we find

$$\begin{aligned} \bar{T}_0^f(\tau, X) = & \sigma\mu C_0(\tau) \left\{ \int_{X_n}^X \bar{V}(\tau, X') dX' - X \int_{X_n}^{X_n+1} \bar{V}(\tau, X') dX' \right. \\ & \left. - \int_{X_n}^{X_n+1} \left[\int_{X_n}^{X'} \bar{V}(\tau, X'') dX'' \right] dX' + \left(X_n + \frac{1}{2} \right) \int_{X_n}^{X_n+1} \bar{V}(\tau, X') dX' \right\}. \end{aligned} \quad (4.11)$$

Equations (4.10) and (4.11) make up a set of coupled nonlinear equations that completely define the mean temperature, \bar{T}_0^m , and the fluctuation, \bar{T}_0^f . The determination of these quantities allows for calculating the temperature field, $\bar{T}_0 = \bar{T}_0^m + \bar{T}_0^f$, the constants $C_0(\tau)$ and $C_1(\tau)$, and thus, by virtue of equation (4.3), the probability density distribution, $\bar{\rho}_0$. However, since the quantities $\bar{\rho}_0$ and $C_0(\tau)$ (that feature in equations (4.10) and (4.11)) depend on temperature in a very complicated way, solving equations (4.10) and (4.11) is not an easy task.

This problem can be strongly simplified in the case in which the fields $\bar{\rho}_0$ and \bar{T}_0 describe the equilibrium state of the system, provided that such a state is a solution of equations (3.10) respectful of boundary conditions. The simplification is because, in the case of equilibrium, temperature \bar{T}_0 is independent of the space coordinate, X , and equals its *mean value*, \bar{T}_0^m , the probability density distribution, $\bar{\rho}_0$, equals the Gibbs distribution, and, consequently, the zeroth-order current is identically zero (i.e. $C_0(\tau) = 0$). Based on this argument, we propose here to approximate the actual zeroth-order temperature field with its mean value, i.e.

$$\bar{T}_0(\tau, X) = \bar{T}_0^m(\tau). \tag{4.12}$$

This implies that the zeroth-order probability density distribution reduces to the Gibbs distribution

$$\bar{\rho}_0(\tau, X) = \frac{1}{Z_-(\tau)} \gamma_-(\tau, X), \tag{4.13}$$

where the quantity $Z_-(\tau)$, which is said to be the *partition function* of the system, is defined through the normalization condition, i.e.

$$Z_-(\tau) = \frac{1}{C_1(\tau)} = \int_{X_n}^{X_{n+1}} \gamma_-(\tau, X') dX'. \tag{4.14}$$

Since, according to equation (4.12), temperature (which, from here on, will be simply denoted by $\bar{T}_0(\tau)$) is independent of space coordinates, the integrand featuring in the exponential in equation (4.5) is exactly integrable, and the functions $\gamma_{\mp}(\tau, X)$ can be thus written as

$$\gamma_{\mp}(\tau, X) = \exp \left[\mp \frac{\phi_c}{\bar{T}_0(\tau)} [\bar{V}(\tau, X) - \bar{V}(\tau, X_n)] \right]. \tag{4.15}$$

We remark that the functions defined in (4.12) and (4.13) satisfy equations (3.10) together with the boundary conditions (3.13) and (3.14). Moreover, since \bar{T}_0 is uniformly distributed in space, it cannot generate any thermal gradient. This result agrees with the fact that thermal current has to be absent at the equilibrium state of the system.

Although the approximation introduced in equation (4.12) (which is remnant of the *mean field approximation*) may be poor for some applications, we use it because it enables us to find the limit in which Parrondo's results [14] can be retrieved. Indeed, in Parrondo's paper, the first addend of expansion of the field ρ (which coincides with our $\bar{\rho}_0$) equals the Gibbs equilibrium distribution as a consequence of the fact that, in that problem, temperature is regarded as a constant, while the non-equilibrium term, denoted by φ , is responsible for generating a current of Brownian particles across the boundary of the lattice reference cell. Hence, in order to re-obtain this picture in the case of temperature variable in time and space, either do we admit that the zeroth-order temperature, \bar{T}_0 , depends on space coordinates and $\bar{\rho}_0$ consists of both an equilibrium term and a non-equilibrium term (the latter being related to the zeroth-order temperature fluctuation \bar{T}_0^f), or we simply approximate the zeroth-order temperature with its mean value, and find that $\bar{\rho}_0$ becomes the Gibbs equilibrium distribution. In this paper, we followed the second path and, by identifying the first-order fields, \bar{T}_1 and $\bar{\rho}_1$, as non-equilibrium corrections, we showed that the current of Brownian particles exhibits a contribution (which was absent in Parrondo's paper) related to the thermal gradient generated by \bar{T}_1 .

4.2. Determination of temperature \bar{T}_0 , and the first-order fields

In order to determine the first-order probability density distribution, $\bar{\rho}_1$, and temperature \bar{T}_1 , we need to solve equations (3.11).

4.2.1. Determination of the first-order probability density distribution. The explicit form of the first-order probability density distribution, $\bar{\rho}_1$, is given by the general solution to equation (3.11), i.e.

$$\begin{aligned} \bar{\rho}_1(\tau, X) = & K_2(\tau)\gamma_-(\tau, X) + K_1(\tau)\frac{\gamma_-(\tau, X)}{\bar{T}_0(\tau)} \int_{X_n}^X \gamma_+(\tau, X') dX' \\ & + \frac{1}{\sigma\bar{T}_0(\tau)}\gamma_-(\tau, X) \int_{X_n}^X \gamma_+(\tau, X') \left(\int_{X_n}^{X'} \frac{\partial\bar{\rho}_0}{\partial\tau}(\tau, X'') dX'' \right) dX' \\ & + \frac{\phi_c}{[\bar{T}_0(\tau)]^2}\bar{\rho}_0(\tau, X) \int_{X_n}^X \bar{T}_1(\tau, X') \frac{\partial\bar{V}}{\partial X}(\tau, X') dX', \end{aligned} \quad (4.16)$$

where

$$\gamma_+(\tau, X) = \exp\left[+\phi_c \frac{\bar{V}(\tau, X)}{\bar{T}_0(\tau)}\right] \quad \text{and} \quad Z_+(\tau) = \int_{X_n}^{X_n+1} \gamma_+(\tau, X) dX. \quad (4.17)$$

According to equations (3.13), the constants $K_1(\tau)$ and $K_2(\tau)$ are calculated by requiring $\bar{\rho}_1$ to satisfy periodic boundary conditions, and imposing that the integral of $\bar{\rho}_1$ over the interval $[X_n, X_n + 1]$ is zero [14], i.e.

$$\bar{\rho}_1(\tau, X_n) = \bar{\rho}_1(\tau, X_n + 1) \quad \text{and} \quad \int_{X_n}^{X_n+1} \bar{\rho}_1(\tau, X) dX = 0. \quad (4.18)$$

By enforcing equations (4.18) and introducing the notation

$$\begin{aligned} f(\tau, X) = & \frac{\gamma_-(\tau, X)}{\sigma\bar{T}_0(\tau)} \left\{ \left[\int_{X_n}^{X_n+1} \gamma_+(\tau, X) \left(\int_{X_n}^X \frac{\partial\bar{\rho}_0}{\partial\tau}(\tau, X') dX' \right) dX \right] \right. \\ & \times \left[\int_{X_n}^{X_n+1} \bar{\rho}_0(\tau, X) \left(\int_{X_n}^X \frac{\gamma_+(\tau, X')}{Z_+(\tau)} dX' \right) dX \right] \\ & - \int_{X_n}^{X_n+1} \bar{\rho}_0(\tau, X) \left(\int_{X_n}^X \gamma_+(\tau, X') \left(\int_{X_n}^{X'} \frac{\partial\bar{\rho}_0}{\partial\tau}(\tau, X'') dX'' \right) dX' \right) dX \\ & + \int_{X_n}^X \gamma_+(\tau, X') \left(\int_{X_n}^{X'} \frac{\partial\bar{\rho}_0}{\partial\tau}(\tau, X'') dX'' \right) dX' \\ & \left. - \left[\int_{X_n}^{X_n+1} \gamma_+(\tau, X) \left(\int_{X_n}^X \frac{\partial\bar{\rho}_0}{\partial\tau}(\tau, X') dX' \right) dX \right] \left[\int_{X_n}^X \frac{\gamma_+(\tau, X')}{Z_+(\tau)} dX' \right] \right\}, \end{aligned} \quad (4.19)$$

it can be proven that the first-order probability density distribution is given by

$$\begin{aligned} \bar{\rho}_1(\tau, X) = & f(\tau, X) + \frac{\phi_c\bar{\rho}_0(\tau, X)}{[\bar{T}_0(\tau)]^2} \left(\int_{X_n}^{X_n+1} \bar{T}_1(\tau, X) \frac{\partial\bar{V}}{\partial X}(\tau, X) dX \right) \\ & \times \left[\int_{X_n}^{X_n+1} \bar{\rho}_0(\tau, X) \left(\int_{X_n}^X \frac{\gamma_+(\tau, X')}{Z_+(\tau)} dX' \right) dX - \int_{X_n}^X \frac{\gamma_+(\tau, X')}{Z_+(\tau)} dX' \right] \end{aligned}$$

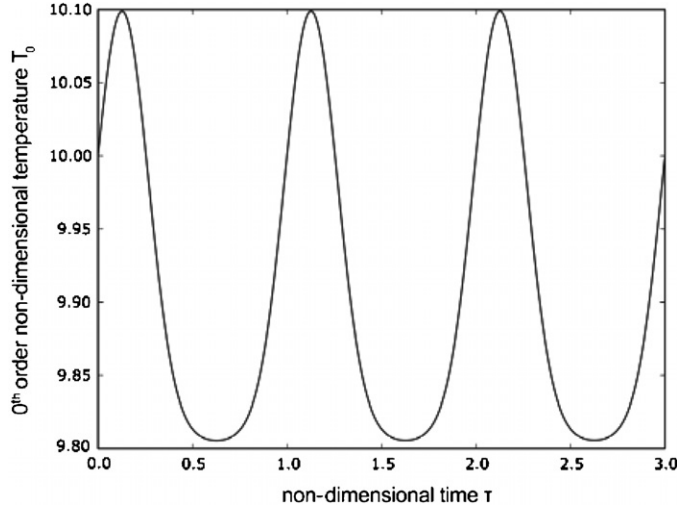


Figure 2. Zeroth-order temperature, \bar{T}_0 , is plotted against nondimensional time $\tau = t/t_V$ over three cycles. This result is obtained by setting $\tau_{in} = 0$ and imposing the condition $A_0 = \bar{T}^{in} = 10$.

$$\begin{aligned}
 & + \frac{\phi_c \bar{\rho}_0(\tau, X)}{[\bar{T}_0(\tau)]^2} \left[\int_{X_n}^X \bar{T}_1(\tau, X') \frac{\partial \bar{V}}{\partial X}(\tau, X') dX' \right. \\
 & \left. - \int_{X_n}^{X_{n+1}} \bar{\rho}_0(\tau, X) \left(\int_{X_n}^X \bar{T}_1(\tau, X') \frac{\partial \bar{V}}{\partial X}(\tau, X') dX' \right) dX \right]. \tag{4.20}
 \end{aligned}$$

4.2.2. *Determination of the zeroth-order temperature.* By employing equation (4.20), we can rewrite equation (3.11) as

$$\frac{\partial \bar{T}_0}{\partial \tau}(\tau) = \frac{\partial^2 \bar{T}_1}{\partial X^2}(\tau, X) + \sigma \mu K_1(\tau) \frac{\partial \bar{V}}{\partial X}(\tau, X) + \mu \frac{\partial \bar{V}}{\partial X}(\tau, X) \int_{X_n}^X \frac{\partial \bar{\rho}_0}{\partial \tau}(\tau, X') dX'. \tag{4.21}$$

Even though equation (4.21) involves \bar{T}_0 , \bar{T}_1 and $\bar{\rho}_0$, it is possible to find a relation which enables us to determine \bar{T}_0 in a self-consistent fashion. Indeed, by integrating equation (4.21) over the unit cell $[X_n, X_{n+1}]$ and appealing for the periodic boundary conditions imposed on the temperature gradient, $\partial \bar{T}_1 / \partial X$, we obtain

$$\frac{d\bar{T}_0}{d\tau}(\tau) = -\mu \int_{X_n}^{X_{n+1}} \bar{V}(\tau, X) \frac{\partial \bar{\rho}_0}{\partial \tau}(\tau, X) dX. \tag{4.22}$$

Since $\bar{\rho}_0$ depends on \bar{T}_0 (cf equations (4.13) and (4.14)), equation (4.22) is a nonlinear integro-differential equation. By integrating both sides of equation (4.22), we find

$$\bar{T}_0(\tau) = A_0 - \mu \int_{\tau_{in}}^{\tau} \left[\int_{X_n}^{X_{n+1}} \bar{V}(\tau', X) \frac{\partial \bar{\rho}_0}{\partial \tau}(\tau', X) dX \right] d\tau', \tag{4.23}$$

where, for the sake of consistency, the constant A_0 has to be taken equal to the initial temperature distribution (i.e. $A_0 = \bar{T}^{in}$). We solved equation (4.22) numerically and, consistently with the time periodicity of the potential, we found that \bar{T}_0 is periodic in time and has the same period of the potential. The result of our calculation is shown in figure 2, where we have set $\bar{T}^{in} = 10$.

We would like to remark that the time periodicity of \bar{T}_0 , although following from the periodicity of the potential and being determined by solving a self-consistency relation, may incorporate the model proposed by Reimann *et al* [19], in which, the authors study the problem of Brownian motors driven by temperature oscillations and evolving in a one-dimensional periodic lattice in the presence of a potential that is periodic in space and independent of time.

4.2.3. Determination of the first-order temperature. Once temperature \bar{T}_0 has been found, the zeroth-order probability density distribution, $\bar{\rho}_0$, can be evaluated by using equations (4.13) and (4.14). The behavior of $\bar{\rho}_0$ is shown in figure 3. We notice that $\bar{\rho}_0$ is a periodic function of time (cf figure 3(b)) and space (cf figure 3(c))

The determination of $\bar{\rho}_0$ and \bar{T}_0 enables us to calculate the first-order temperature, \bar{T}_1 . In order to do that, we first rewrite equation (4.21) as

$$\frac{\partial^2 \bar{T}_1}{\partial X^2}(\tau, X) = \frac{\partial \bar{T}_0}{\partial \tau}(\tau) - \sigma \mu \frac{\partial \bar{V}}{\partial X}(\tau, X) \left[K_1(\tau) + \frac{1}{\sigma} \int_{X_n}^X \frac{\partial \bar{\rho}_0}{\partial \tau}(\tau, X') dX' \right], \quad (4.24)$$

and determine the constant $K_1(\tau)$ by invoking the first condition of equations (4.18), i.e.

$$K_1(\tau) = - \frac{\phi_c}{Z_+(\tau)Z_-(\tau)\bar{T}_0(\tau)} \int_{X_n}^{X_{n+1}} \bar{T}_1(\tau, X) \frac{\partial \bar{V}}{\partial X}(\tau, X) dX - \frac{1}{\sigma} \int_{X_n}^{X_{n+1}} \frac{\gamma_+(\tau, X)}{Z_+(\tau)} \left(\int_{X_n}^X \frac{\partial \bar{\rho}_0}{\partial \tau}(\tau, X') dX' \right) dX. \quad (4.25)$$

By substituting equation (4.25) into equation (4.24) and introducing the auxiliary function

$$G(\tau, X) = \frac{\partial \bar{T}_0}{\partial \tau}(\tau) + \mu \frac{\partial \bar{V}}{\partial X}(\tau, X) \times \left[\int_{X_n}^{X_{n+1}} \frac{\gamma_+(t, X)}{Z_+(\tau)} \left(\int_{X_n}^X \frac{\partial \bar{\rho}_0}{\partial \tau}(\tau, X') dX' \right) dX - \int_{X_n}^X \frac{\partial \bar{\rho}_0}{\partial \tau}(\tau, X') dX' \right], \quad (4.26)$$

we rearrange equation (4.24) as

$$\frac{\partial^2 \bar{T}_1}{\partial X^2}(\tau, X) = G(\tau, X) + \kappa(\tau) \frac{\partial \bar{V}}{\partial X}(\tau, X) \int_{X_n}^{X_{n+1}} \bar{T}_1(\tau, X) \frac{\partial \bar{V}}{\partial X}(\tau, X) dX, \quad (4.27)$$

where

$$\kappa(\tau) = \frac{\mu \sigma \phi_c}{Z_-(\tau)Z_+(\tau)\bar{T}_0(\tau)}. \quad (4.28)$$

Since the potential is assumed to be differentiable in the interval $[X_n, X_n + 1]$, equation (4.27) can be formally solved by integrating twice, i.e.

$$\bar{T}_1(\tau, X) = \bar{T}_1(\tau, X_n) + \left[\frac{\partial \bar{T}_1}{\partial X}(\tau, X_n) \right] (X - X_n) + \int_{X_n}^X \left(\int_{X_n}^{X'} G(\tau, X'') dX'' \right) dX' + \kappa(\tau) \left[\int_{X_n}^{X_{n+1}} \bar{T}_1(\tau, X) \frac{\partial \bar{V}}{\partial X}(\tau, X) dX \right] \int_{X_n}^X \bar{V}(\tau, X') dX'. \quad (4.29)$$

The enforcement of periodic boundary conditions on temperature \bar{T}_1 allows for determining the value of the temperature gradient at the lattice site, X_n . Indeed, by introducing the notation

$$Q(\tau, X) = \frac{\partial \bar{T}_1}{\partial X}(\tau, X), \quad (4.30)$$

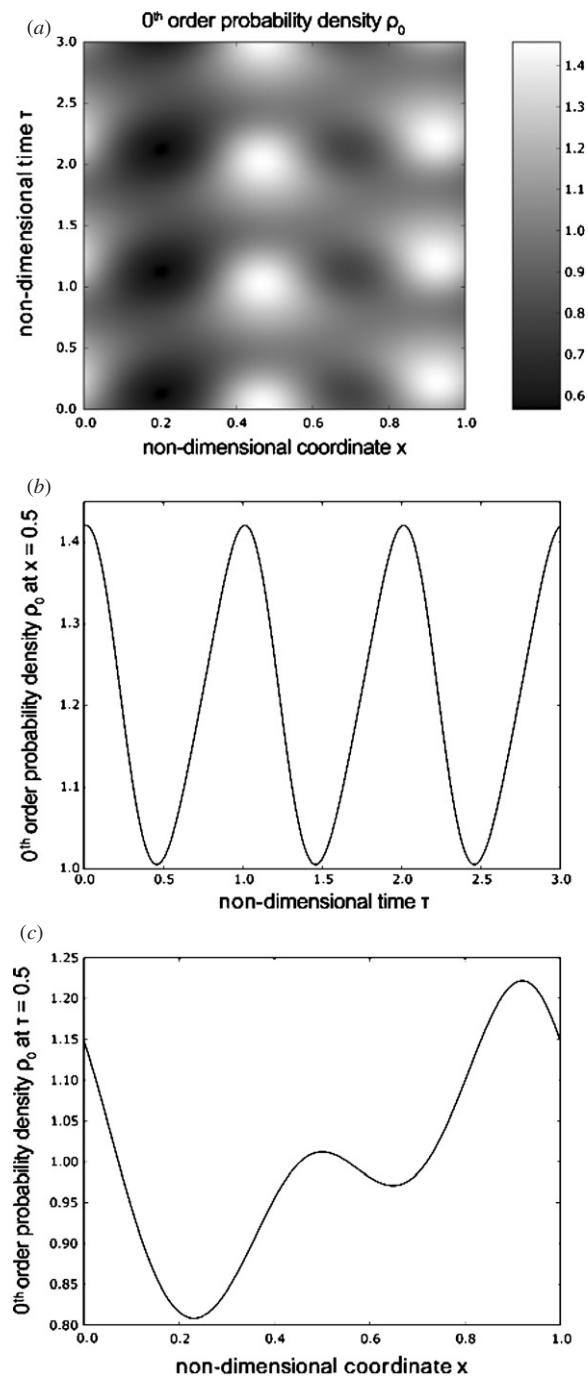


Figure 3. Gibbs probability density distribution, $\bar{\rho}_0$. (a) $\bar{\rho}_0$ is plotted against nondimensional time $\tau = t/t_V$ over three cycles (i.e. $\tau \in [0, 3]$) and space $X \in [X_n, X_n + 1]$. (b) $\bar{\rho}_0$ is evaluated at $X = 0.5$ and plotted against time. (c) $\bar{\rho}_0$ is evaluated at time $t = 0.5t_V$ (i.e. $\tau = 0.5$) and plotted against space.

the temperature gradient at the lattice site X_n , $Q(\tau, X_n)$, is given by

$$Q(\tau, X_n) = - \int_{X_n}^{X_{n+1}} \left(\int_{X_n}^X G(\tau, X') dX' \right) dX - \kappa(\tau) \left[\int_{X_n}^{X_{n+1}} \bar{T}_1(\tau, X'') \frac{\partial \bar{V}}{\partial X}(\tau, X'') dX'' \right] \int_{X_n}^{X_{n+1}} \bar{V}(\tau, X) dX. \quad (4.31)$$

By integrating equation (4.27) and substituting equation (4.31) into the resulting expression, we find

$$Q(\tau, X) = F(\tau, X) + \kappa(\tau)U(\tau, X) \int_{X_n}^{X_{n+1}} \bar{T}_1(\tau, X') \frac{\partial \bar{V}}{\partial X}(\tau, X') dX', \quad (4.32)$$

where

$$F(\tau, X) = \int_{X_n}^X G(\tau, X') dX' - \int_{X_n}^{X_{n+1}} \left(\int_{X_n}^X G(\tau, X') dX' \right) dX, \quad (4.33)$$

and

$$U(\tau, X) = \bar{V}(\tau, X) - \int_{X_n}^{X_{n+1}} \bar{V}(\tau, X') dX'. \quad (4.34)$$

We note that temperature \bar{T}_1 can be eliminated from equation (4.32) by invoking the identity

$$\int_{X_n}^{X_{n+1}} \bar{T}_1(\tau, X) \frac{\partial \bar{V}}{\partial X}(\tau, X) dX = - \int_{X_n}^{X_{n+1}} \bar{V}(\tau, X) Q(\tau, X) dX, \quad (4.35)$$

which holds true because \bar{T}_1 has to satisfy the periodic boundary condition $\bar{T}_1(\tau, X_n) = \bar{T}_1(\tau, X_{n+1})$. By virtue of equation (4.35), equation (4.32) can be written as

$$Q(\tau, X) = F(\tau, X) - \kappa(\tau)U(\tau, X) \int_{X_n}^{X_{n+1}} Q(\tau, X') \bar{V}(\tau, X') dX'. \quad (4.36)$$

This equation enables us to determine the temperature gradient, $Q(\tau, X)$, and to calculate temperature \bar{T}_1 .

Equation (4.36) is a Fredholm integral equation of second kind with degenerate kernel $G(\tau, X, X') = U(\tau, X)\bar{V}(\tau, X')$. Its solution is given by [20]

$$Q(\tau, X) = F(\tau, X) - U(\tau, X) \frac{\kappa(\tau) \int_{X_n}^{X_{n+1}} \bar{V}(\tau, X) F(\tau, X) dX}{1 + \kappa(\tau) \int_{X_n}^{X_{n+1}} \bar{V}(\tau, X) U(\tau, X) dX}. \quad (4.37)$$

Since $Q(\tau, X)$ is the temperature gradient, \bar{T}_1 is found by direct integration, i.e.

$$\bar{T}_1(\tau, X) = A(\tau) + P(\tau, X), \quad (4.38)$$

where the convention

$$A(\tau) = \bar{T}_1(\tau, X_n) \quad \text{and} \quad P(\tau, X) = \int_{X_n}^X Q(\tau, X') dX' \quad (4.39)$$

has been used for notational convenience.

4.2.4. *Determination of the constant $A(\tau)$.* In order for temperature \bar{T}_1 to be completely determined, we need to calculate the constant $A(\tau)$. This is done by imposing that the result given in equation (4.38) is consistent with the second equation of the set (3.12). By virtue of consistency, the integral on the left-hand side of this equation over the interval $[X_n, X_n + 1]$ has to be equal to the integral on the right-hand side over the same interval. Since the gradient $\partial \bar{T}_2 / \partial X$ is a periodic function of space, its integral over the reference lattice cell is identically zero, and the requirement above leads to the condition

$$\frac{d}{d\tau} \int_{X_n}^{X_{n+1}} \bar{T}_1(\tau, X) dX = \sigma \mu \int_{X_n}^{X_{n+1}} \frac{\partial \bar{V}}{\partial X}(\tau, X) \left[\bar{T}_1(\tau, X) \frac{\partial \bar{\rho}_1}{\partial X}(\tau, X) \right] dX. \quad (4.40)$$

Since potential \bar{V} , temperature \bar{T}_1 and probability density distribution $\bar{\rho}_1$ are all periodic in space, integration by parts of the right-hand side of equation (4.40) yields

$$\frac{d}{d\tau} \int_{X_n}^{X_{n+1}} \bar{T}_1(\tau, X) dX = -\sigma \mu \int_{X_n}^{X_{n+1}} \bar{V}(\tau, X) \frac{\partial}{\partial X} \left[\bar{T}_1(\tau, X) \frac{\partial \bar{\rho}_1}{\partial X}(\tau, X) \right] dX. \quad (4.41)$$

Furthermore, by using the first equation of the set (3.12) and substituting the expression of \bar{T}_1 given in equation (4.38) into the left-hand side term of equation (4.41), we obtain

$$\frac{dA}{d\tau}(\tau) + \frac{d}{d\tau} \int_{X_n}^{X_{n+1}} P(\tau, X) dX = -\mu \int_{X_n}^{X_{n+1}} \bar{V}(\tau, X) \frac{\partial \bar{\rho}_1}{\partial \tau}(\tau, X) dX. \quad (4.42)$$

In order to highlight the dependence of $\bar{\rho}_1$ on $A(\tau)$ and $P(\tau, X)$, it is convenient to determine an expression in which all the terms featuring $P(\tau, X)$ are separated from the terms multiplied by $A(\tau)$. After some algebraic manipulations, this expression is found to be

$$\bar{\rho}_1(\tau, X) = h(\tau, X) + \frac{\phi_c \bar{\rho}_0(\tau, X)}{[\bar{T}_0(\tau)]^2} \tilde{V}(\tau, X) A(\tau), \quad (4.43)$$

where

$$h(\tau, X) = f(\tau, X) + g(\tau, X), \quad (4.44)$$

$$\tilde{V}(\tau, X) = V(\tau, X) - \int_{X_n}^{X_{n+1}} \bar{\rho}_0(\tau, X) \bar{V}(\tau, X) dX \quad (4.45)$$

and $g(\tau, X)$ denotes the auxiliary function

$$g(\tau, X) = \frac{\phi_c \bar{\rho}_0(\tau, X)}{[\bar{T}_0(\tau)]^2} \left\{ \left[\int_{X_n}^X \frac{\gamma_+(\tau, X')}{Z_+(\tau)} dX' - \int_{X_n}^{X_{n+1}} \bar{\rho}_0(\tau, X) \left(\int_{X_n}^{X_{n+1}} \frac{\gamma_+(\tau, X')}{Z_+(\tau)} dX' \right) dX \right] \right. \\ \times \left[\int_{X_n}^{X_{n+1}} Q(\tau, X) \bar{V}(\tau, X) dX \right] + \int_{X_n}^X P(\tau, X') \frac{\partial \bar{V}}{\partial X}(\tau, X') dX' \\ \left. - \int_{X_n}^{X_{n+1}} \bar{\rho}_0(\tau, X) \left(\int_{X_n}^X P(\tau, X') \frac{\partial \bar{V}}{\partial X}(\tau, X') dX' \right) dX \right\}. \quad (4.46)$$

By substituting equation (4.43) into equation (4.42), we find the self-consistent differential equation which allows for determining the constant $A(\tau)$, i.e.

$$\frac{dA}{d\tau}(\tau) + N(\tau)A(\tau) = -I(\tau), \quad (4.47)$$

where we introduced the compact notation

$$I(\tau) = \frac{\mu \int_{X_n}^{X_{n+1}} \bar{V}(\tau, X) \frac{\partial h}{\partial \tau}(\tau, X) dX + \frac{d}{d\tau} \int_{X_n}^{X_{n+1}} P(\tau, X) dX}{1 + \frac{\mu \phi_c}{[\bar{T}_0(\tau)]^2} \int_{X_n}^{X_{n+1}} \bar{\rho}_0(\tau, X) \bar{V}(\tau, X) \tilde{V}(\tau, X) dX} \quad (4.48)$$

and

$$\begin{aligned}
 N(\tau) = & \frac{1}{1 + \frac{\mu\phi_c}{[\bar{T}_0(\tau)]^2} \int_{X_n}^{X_{n+1}} \bar{\rho}_0(\tau, X) \bar{V}(\tau, X) \tilde{V}(\tau, X) dX} \\
 & \times \left\{ \frac{\mu\phi_c}{[\bar{T}_0(\tau)]^2} \left[\int_{X_n}^{X_{n+1}} \bar{V}(\tau, X) \frac{\partial \bar{\rho}_0}{\partial \tau}(\tau, X) \tilde{V}(\tau, X) dX \right. \right. \\
 & \left. \left. + \int_{X_n}^{X_{n+1}} \bar{V}(\tau, X) \bar{\rho}_0(\tau, X) \frac{\partial \tilde{V}}{\partial \tau}(\tau, X) dX \right] \right. \\
 & \left. - \frac{2\mu\phi_c}{[\bar{T}_0(\tau)]^3} \frac{\partial \bar{T}_0}{\partial \tau}(\tau) \int_{X_n}^{X_{n+1}} \bar{\rho}_0(\tau, X) \bar{V}(\tau, X) \tilde{V}(\tau, X) dX \right\}. \quad (4.49)
 \end{aligned}$$

By imposing the initial condition $A(\tau_{in}) = 0$, the solution to equation (4.47) is given by

$$A(\tau) = - \left[\int_{\tau_{in}}^{\tau} I(\tau') e^{\int_{\tau_{in}}^{\tau'} N(\tau'') d\tau''} d\tau' \right] e^{-\int_{\tau_{in}}^{\tau} N(\tau') d\tau'}. \quad (4.50)$$

The constant $A(\tau)$ is found by evaluating the right-hand side of equation (4.50) numerically. Finally, substitution into equation (4.38) allows for obtaining temperature \bar{T}_1 . This enables us to completely determine the first-order probability density distribution $\bar{\rho}_1$.

The results of the calculation of \bar{T}_1 and $\bar{\rho}_1$ are shown in figures 4 and 5, respectively, for three different values of the parameter σ . We evaluated \bar{T}_1 and $\bar{\rho}_1$ for $\sigma = 1$, $\sigma = 0.1$ and $\sigma = 0.01$.

5. Directed transport of Brownian particles

In this section, we show how directed transport of Brownian particles takes place in the presence of a changing temperature field. In general, the existence of directed transport of Brownian particles is proven by determining the net current of particles crossing the boundary of the lattice cell $[X_n, X_{n+1}]$. By invoking the asymptotic expansion of the probability current given in equation (3.7) and truncating the expansion at the first order, we obtain

$$J \approx \varepsilon \frac{\rho_c k_B T_c}{L\zeta} \bar{J}_1 = -\varepsilon \frac{\rho_c k_B T_c}{L\zeta} \left[\bar{T}_0 \frac{\partial \bar{\rho}_1}{\partial X} + \bar{T}_1 \frac{\partial \bar{\rho}_0}{\partial X} + \phi_c \frac{\partial \bar{V}}{\partial X} \bar{\rho}_1 \right]. \quad (5.1)$$

We note that the zeroth-order current, \bar{J}_0 (cf equations (3.8)), vanishes identically because the Gibbs probability distribution, $\bar{\rho}_0$, describes a zero-current state.

By substituting the expression of $\bar{\rho}_1$ given in equation (4.16) into equation (5.1), we find

$$\bar{J}_1(\tau, X) = - \left[K_1(\tau) + \frac{1}{\sigma} \int_{X_n}^X \frac{\partial \bar{\rho}_0}{\partial \tau}(\tau, X') dX' \right]. \quad (5.2)$$

Moreover, by appealing to the definition of the constant $K_1(\tau)$ in equation (4.25), we can rearrange \bar{J}_1 as

$$\begin{aligned}
 \bar{J}_1(\tau, X) = & \frac{1}{\sigma} \int_{X_n}^{X_{n+1}} \frac{\gamma_+(\tau, X')}{Z_+(\tau)} \left(\int_{X_n}^{X'} \frac{\partial \bar{\rho}_0}{\partial \tau}(\tau, X'') dX'' \right) dX' + \frac{\phi_c}{Z_-(\tau)Z_+(\tau)\bar{T}_0(\tau)} \\
 & \times \int_{X_n}^{X_{n+1}} \bar{T}_1(\tau, X') \frac{\partial \bar{V}}{\partial X}(\tau, X') dX' - \frac{1}{\sigma} \int_{X_n}^X \frac{\partial \bar{\rho}_0}{\partial \tau}(\tau, X') dX'. \quad (5.3)
 \end{aligned}$$

Since we are interested in calculating the current of particles crossing the boundary of the system, we need to evaluate \bar{J}_1 at the lattice sites X_n and $X_n + 1$. By noting that the normalization condition on $\bar{\rho}_0$ implies

$$0 = \int_{X_n}^{X_{n+1}} \frac{\partial \bar{\rho}_0}{\partial \tau}(\tau, X') dX' = \frac{d}{d\tau} \int_{X_n}^{X_{n+1}} \bar{\rho}_0(\tau, X') dX', \quad (5.4)$$

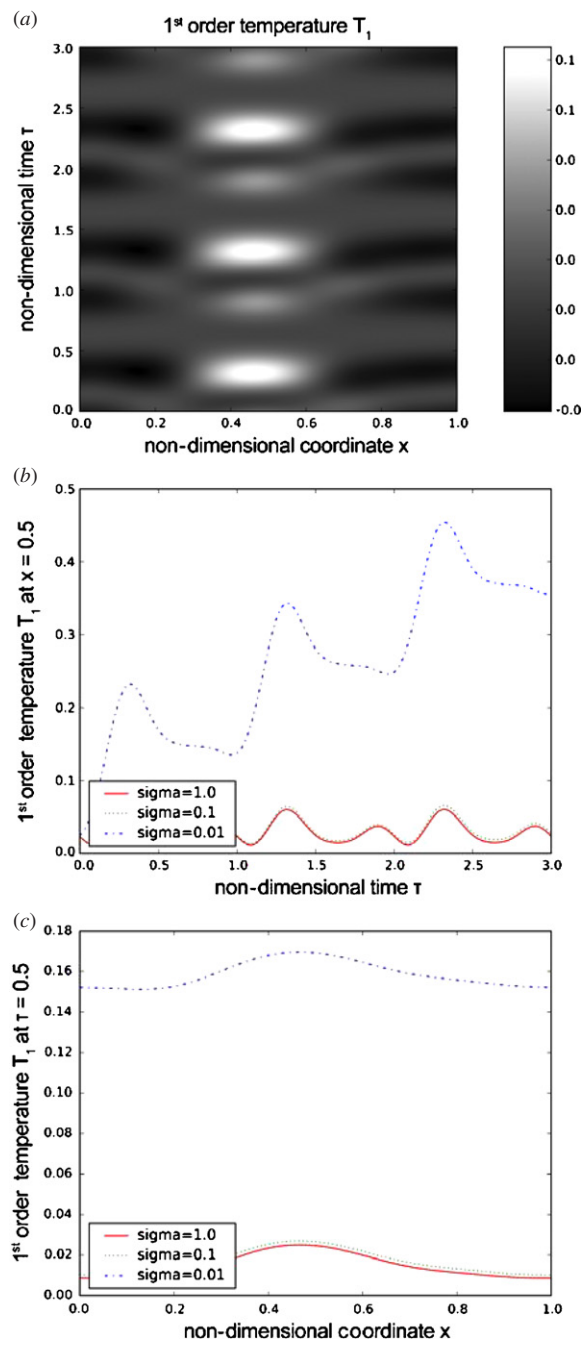


Figure 4. First-order temperature, \bar{T}_1 . (a) \bar{T}_1 is plotted against nondimensional time $\tau = t/t_V$ over three cycles (i.e. $\tau \in [0, 3]$) and space $X \in [X_n, X_n + 1]$. (b) \bar{T}_1 is evaluated at $X = 0.5$ and plotted against time. (c) \bar{T}_1 is evaluated at time $t = 0.5t_V$ (i.e. $\tau = 0.5$) and plotted against space.

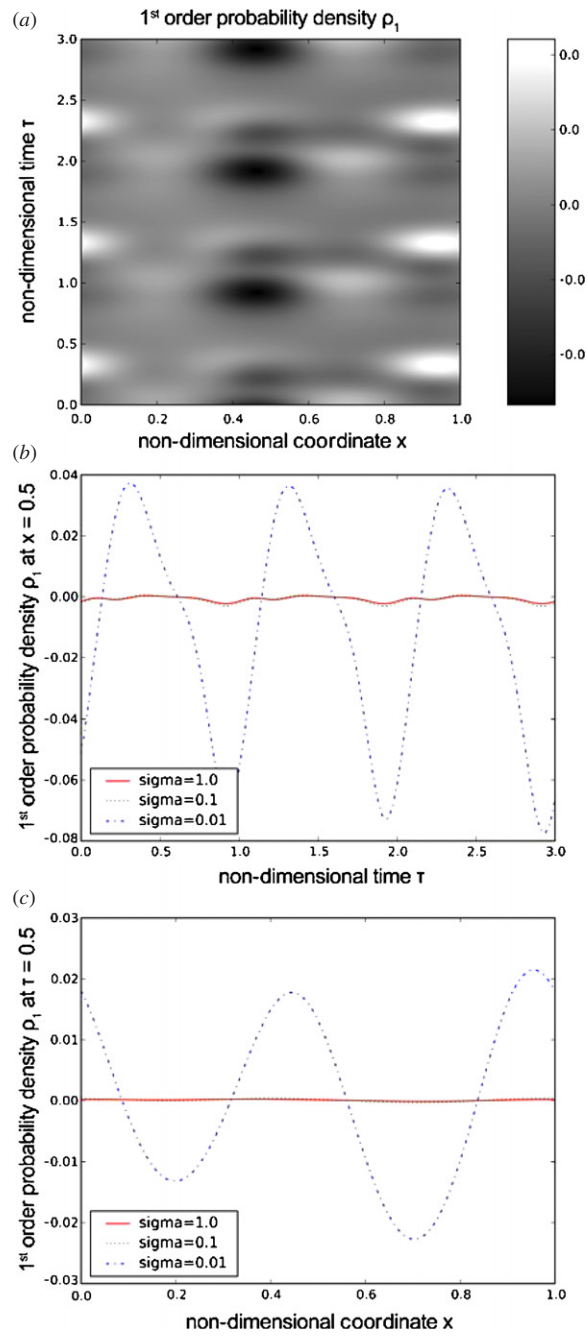


Figure 5. First-order probability density distribution, $\bar{\rho}_1$. (a) $\bar{\rho}_1$ is plotted against nondimensional time $\tau = t/t_V$ over three cycles (i.e. $\tau \in [0, 3]$) and space $X \in [X_n, X_n + 1]$. (b) $\bar{\rho}_1$ is evaluated at $X = 0.5$ and plotted against time. (c) $\bar{\rho}_1$ is evaluated at time $t = 0.5t_V$ (i.e. $\tau = 0.5$) and plotted against space.

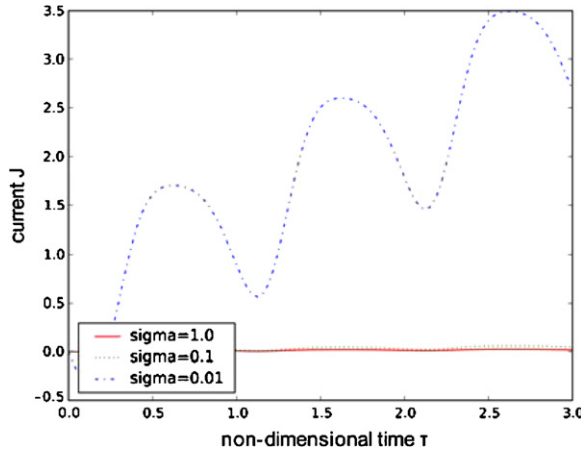


Figure 6. Global current of Brownian particles crossing the boundary of the lattice cell $[X_n, X_n + 1]$

and rewriting the second term on the right-hand side of equation (5.3) as

$$\int_{X_n}^{X_{n+1}} \bar{T}_1(\tau, X') \frac{\partial \bar{V}}{\partial X}(\tau, X') dX' = - \int_{X_n}^{X_{n+1}} \bar{V}(\tau, X') \frac{\partial \bar{T}_1}{\partial X}(\tau, X') dX', \quad (5.5)$$

we obtain

$$\begin{aligned} \bar{J}_1(\tau, X_n) = \bar{J}_1(\tau, X_{n+1}) = & \frac{1}{\sigma} \int_{X_n}^{X_{n+1}} \frac{\gamma_+(\tau, X')}{Z_+(\tau)} \left(\int_{X_n}^{X'} \frac{\partial \bar{\rho}_0}{\partial \tau}(\tau, X'') dX'' \right) dX' \\ & - \frac{\phi_c}{Z_-(\tau)Z_+(\tau)\bar{T}_0(\tau)} \int_{X_n}^{X_{n+1}} \bar{V}(\tau, X') \frac{\partial \bar{T}_1}{\partial X}(\tau, X') dX'. \end{aligned} \quad (5.6)$$

The probability current given in equation (5.6) is periodic in space. The global current of Brownian particles accumulated after k adiabatic cycles of the potential is found by integrating equation (5.6) over the time interval $[\tau_{in}, \tau_{in} + k]$, i.e.

$$\begin{aligned} \bar{J}_1(\tau_{in}, k) = & \int_{\tau_{in}}^{\tau_{in}+k} \left\{ \frac{1}{\sigma} \int_{X_n}^{X_{n+1}} \frac{\gamma_+(\tau, X')}{Z_+(\tau)} \left(\int_{X_n}^{X'} \frac{\partial \bar{\rho}_0}{\partial \tau}(\tau, X'') dX'' \right) dX' \right\} d\tau \\ & - \int_{\tau_{in}}^{\tau_{in}+k} \left\{ \frac{\phi_c}{Z_-(\tau)Z_+(\tau)\bar{T}_0(\tau)} \int_{X_n}^{X_{n+1}} \bar{V}(\tau, X') \frac{\partial \bar{T}_1}{\partial X}(\tau, X') dX' \right\} d\tau. \end{aligned} \quad (5.7)$$

The result of this calculation is shown in figure 6 for $k = 3$ (i.e. three cycles of the adiabatic potential), and $\sigma = 1$, $\sigma = 0.1$ and $\sigma = 0.01$.

6. Implications for the first and second law of thermodynamics

In this section, we investigate the implications of the theory discussed so far on the first and second law of thermodynamics.

6.1. First law of thermodynamics

Equation (2.10) provides the expression of the *power inserted into the system* described by Streater's model. In nondimensional units, we can rewrite equation (2.10) as

$$\frac{dE}{dt}(t) = \frac{\rho_c V_c L}{t_V} \int_{X_n}^{X_{n+1}} \bar{\rho}(\tau, X) \frac{\partial \bar{V}}{\partial \tau}(\tau, X) dX. \quad (6.1)$$

By appealing for the asymptotic expansion of the probability density distribution given in equations (3.6) and truncating the expansion at the first order, we obtain

$$\frac{dE}{dt}(t) \approx \frac{\rho_c V_c L}{t_V} \left[\int_{X_n}^{X_{n+1}} \bar{\rho}_0(\tau, X) \frac{\partial \bar{V}}{\partial \tau}(\tau, X) dX + \varepsilon \int_{X_n}^{X_{n+1}} \bar{\rho}_1(\tau, X) \frac{\partial \bar{V}}{\partial \tau}(\tau, X) dX \right]. \quad (6.2)$$

Equation (6.2) must be used to evaluate the variation of energy in response to an adiabatic variation of the potential over one or more cycles. In order to do that, it suffices to integrate equation (6.2) over the time interval $[0, t_V]$ (from here on, we set $\tau_{in} = 0$), i.e.

$$E(t_V) - E(0) \approx \rho_c V_c L \left\{ \int_0^1 \left(\int_{X_n}^{X_{n+1}} \bar{\rho}_0(\tau, X) \frac{\partial \bar{V}}{\partial \tau}(\tau, X) dX \right) d\tau + \varepsilon \int_0^1 \left(\int_{X_n}^{X_{n+1}} \bar{\rho}_1(\tau, X) \frac{\partial \bar{V}}{\partial \tau}(\tau, X) dX \right) d\tau \right\}. \quad (6.3)$$

By taking into account equation (4.22) and defining the statistical average of the ratchet potential as

$$\langle \bar{V} \rangle(\tau) = \int_{X_n}^{X_{n+1}} \bar{\rho}_0(\tau, X) \bar{V}(\tau, X) dX, \quad (6.4)$$

integration by parts of the first term on the right-hand side of equation (6.3) leads to

$$E(t_V) - E(0) \approx \rho_c V_c L \left\{ [\langle \bar{V} \rangle(1) - \langle \bar{V} \rangle(0)] + \frac{1}{\mu} [\bar{T}_0(1) - \bar{T}_0(0)] + \varepsilon \int_0^1 \left(\int_{X_n}^{X_{n+1}} \bar{\rho}_1(\tau, X) \frac{\partial \bar{V}}{\partial \tau}(\tau, X) dX \right) d\tau \right\}. \quad (6.5)$$

Since the statistical average of the ratchet potential, $\langle \bar{V} \rangle$, and the zeroth-order temperature, \bar{T}_0 , are periodic functions of time (cf figure 2), the terms between brackets in equation (6.5) are identically zero, and we can write

$$E(t_V) - E(0) \approx \rho_c V_c L \left[\varepsilon \int_0^1 \left(\int_{X_n}^{X_{n+1}} \bar{\rho}_1(\tau, X) \frac{\partial \bar{V}}{\partial \tau}(\tau, X) dX \right) d\tau \right]. \quad (6.6)$$

According to equation (6.6), the energy variation over k cycles is given by

$$E(kt_V) - E(0) \approx \rho_c V_c L \left[\varepsilon \int_0^k \left(\int_{X_n}^{X_{n+1}} \bar{\rho}_1(\tau, X) \frac{\partial \bar{V}}{\partial \tau}(\tau, X) dX \right) d\tau \right]. \quad (6.7)$$

We calculated equation (6.6) numerically for the case $k = 3$, and found that the variation in energy is *not* exactly zero. The result of our calculation is reported in figure 7. While the smallness of energy variation is caused by the multiplication of the right-hand side of equation (6.7) by the smallness parameter ε , the fact that it is not zero is a consequence of the first-order probability density distribution, $\bar{\rho}_1$. Therefore, we conclude that energy variation is a first-order effect.

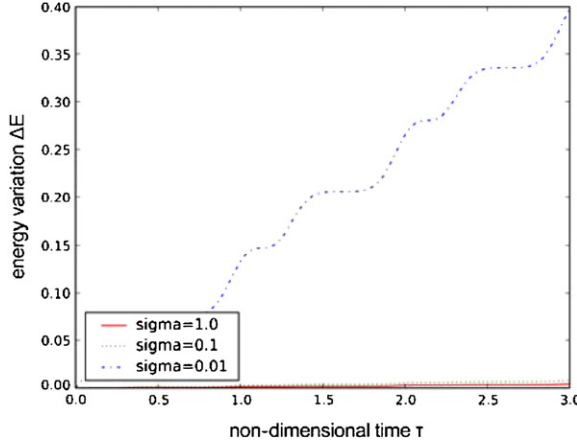


Figure 7. Energy variation over three cycles.

6.2. Second law of thermodynamics

Equation (2.13) states that entropy variation is greater than, or equal to, zero. In nondimensional units, we can write equation (2.13) as

$$\frac{dS}{dt}(t) = \frac{\rho_c k_B^2 T_c}{L \zeta} \int_{X_n}^{X_{n+1}} \frac{[\bar{J}(\tau, X)]^2}{\bar{T}(\tau, X) \bar{\rho}(\tau, X)} dX + \frac{\lambda}{L} \int_{x_n}^{x_n+L} \left[\frac{1}{\bar{T}(\tau, X)} \frac{\partial \bar{T}}{\partial X}(\tau, X) \right]^2 dX \geq 0. \tag{6.8}$$

By substituting the asymptotic expansion of current, \bar{J} , temperature, \bar{T} , and probability density distribution, $\bar{\rho}$, and truncating the expansions at the first-order, we can rearrange equation (6.8) as

$$\begin{aligned} \frac{dS}{dt}(t) = & \frac{\rho_c k_B^2 T_c}{L \zeta} \int_{X_n}^{X_{n+1}} \frac{\varepsilon^2 [\bar{J}_1(\tau, X)]^2}{[\bar{T}_0(\tau) + \varepsilon \bar{T}_1(\tau, X)][\bar{\rho}_0(\tau, X) + \varepsilon \bar{\rho}_1(\tau, X)]} dX \\ & + \frac{\lambda}{L} \int_{x_n}^{x_n+L} \left[\frac{\varepsilon}{\bar{T}_0(\tau) + \varepsilon \bar{T}_1(\tau, X)} \frac{\partial \bar{T}_1}{\partial X}(\tau, X) \right]^2 dX \geq 0. \end{aligned} \tag{6.9}$$

By expanding equation (6.9) in Taylor series in a neighborhood of $\varepsilon = 0$, we obtain

$$\begin{aligned} \frac{dS}{dt}(t) \approx & \varepsilon^2 \frac{\rho_c k_B^2 T_c}{L \zeta} \int_{X_n}^{X_{n+1}} \frac{[\bar{J}_1(\tau, X)]^2}{\bar{T}_0(\tau) \bar{\rho}_0(\tau, X)} dX \\ & + \varepsilon^2 \frac{\lambda}{L} \int_{x_n}^{x_n+L} \left[\frac{1}{\bar{T}_0(\tau)} \frac{\partial \bar{T}_1}{\partial X}(\tau, X) \right]^2 dX \geq 0. \end{aligned} \tag{6.10}$$

This result implies that the theory presented in this paper predicts that entropy variation is a *second-order effect*. Since we limited our description to the first order in ε , we conclude that entropy can be regarded as constant within the adiabatic limit $\varepsilon = t_T/t_V \ll 1$.

7. Conclusions

We studied the interaction of Brownian particles with a changing temperature field in the presence of a one-dimensional, periodic and asymmetric adiabatic potential (cf equation (4.1) and figure 1). Our aim was to show how directed transport takes place and compare our results

with the case of Brownian particles in a thermal bath [14]. In order to investigate the system described above, we had recourse to Streater's model [15, 16], i.e. we studied a set of equations consisting of a Smoluchowski equation and a nonlinear Fourier equation (cf equations (2.1)). We carried out the analysis of Streater's model by performing the asymptotic expansion of equations (2.1). This was done by sorting out two well separated time scales (i.e. the time scale t_T associated with thermal diffusion and the time scale t_V associated with the slow cyclic variation of the adiabatic potential), expanding the probability density distribution, ρ , and temperature, T , in asymptotic series [21, 22] (cf equations (3.6)), and studying the expanded Streater's equations (cf equations (3.10)–(3.12)) by perturbing the system from its thermodynamic equilibrium state.

In order to retrieve, at the lowest order of our perturbative study, the results presented by Parrondo [14], we approximated temperature \bar{T}_0 with its mean value and showed that the zeroth-order probability density distribution, $\bar{\rho}_0$, describes the thermodynamic equilibrium state of the system, while the first-order terms, $\bar{\rho}_1$ and \bar{T}_1 , are non-equilibrium corrections.

We found that the zeroth-order temperature, \bar{T}_0 (cf equation (4.22)) is periodic in time with the same period, t_V , as the ratchet potential (cf figure 2).

The periodicity of \bar{T}_0 implies that Gibbs distribution, $\bar{\rho}_0$, is also periodic in time (cf figure 3(b)). Furthermore, since the potential is assumed to be a periodic function of space, the Gibbs distribution is periodic in space too (cf figure 3(c)).

We remark that, since $\bar{\rho}_0$ and \bar{T}_0 , although descending from a framework more general than that presented in [14], are equilibrium quantities, the influence of a changing temperature field can only be studied by focusing on the first-order (non-equilibrium) fields $\bar{\rho}_1$ and \bar{T}_1 .

By looking at figure 4(b), we notice that \bar{T}_1 is *not* periodic in time. In particular, the deviation from periodicity appears to be irrelevant when σ is of the order of unity (i.e. $\sigma \sim 1$), while it becomes noticeable as σ decreases (e.g. $\sigma = 0.01$). In the latter case, \bar{T}_1 exhibits a marked positive '*drift*', which signals an increment of temperature after a cyclic variation of the adiabatic potential.

In figure 5(b), we showed that the first-order probability density distribution, $\bar{\rho}_1$, is not periodic in time. In this case, the deviation from periodicity is due to a negative '*drift*', which becomes noticeable as σ decreases (e.g. $\sigma = 0.01$).

In figures 4(c) and 5(c), we reported the spatial behavior of \bar{T}_1 and $\bar{\rho}_1$, respectively. These figures show that, by virtue of the periodic boundary conditions imposed on temperature and the first-order probability density distribution, both \bar{T}_1 and $\bar{\rho}_1$ are exactly periodic functions of space.

The first-order terms, $\bar{\rho}_1$ and \bar{T}_1 , are responsible for generating directed transport. In particular, we showed that, since temperature \bar{T}_1 is *not* uniformly distributed, it produces a thermal gradient, which contributes to 'push' Brownian particles across the boundary of the lattice cell $[X_n, X_n + 1]$. Therefore, the global current of Brownian particles consists of two contributions that can be identified with the first and the second addend on the right-hand side of equation (5.7), respectively. The first contribution coincides with the current obtained by Parrondo [14], while the second contribution describes a current due to a thermal gradient, and, as such, cannot feature in the case of Brownian particles in a thermal bath at constant temperature. We conclude that the presence of a thermal gradient (this is a first-order effect according to theory exposed in this paper) is a source of directed transport for Brownian particles. The global current is shown in figure 6. We notice that current exhibits a positive '*drift*', which testifies that the number of particles crossing the boundary of the lattice cell per unit time increases after each cycle.

In this paper we also evaluated the time variation of the global energy of the system (cf equations (6.6) and (6.7)), and we showed that only the zeroth-order contribution vanishes

identically (this is due to the time periodicity of both the statistical average of the potential, $\langle \bar{V} \rangle$ and temperature \bar{T}_0), while the first-order term induces a nonzero energy variation after each cycle (cf figure 7). By comparing equation (6.3) with equation (6.6), we notice that the first integral on the right-hand side of equation (6.3) has to vanish, i.e.

$$\rho_c V_c L \left\{ \int_0^1 \left(\int_{X_n}^{X_{n+1}} \bar{\rho}_0(\tau, X) \frac{\partial \bar{V}}{\partial \tau}(\tau, X) dX \right) d\tau = 0. \right. \quad (7.1)$$

Although the result obtained in equation (7.1) is consistent with Parrondo's description of a Brownian motor in a thermal bath [14], we notice that, in our case, energy variation does not reduce to an exact differential form in the space of potential's parameters. Therefore, whereas in Parrondo's case (i.e. temperature was regarded as a constant), the vanishing of the energy variation led to the definition of *reversible ratchets*, in the case studied in this paper (i.e. temperature is treated as a field changing in space and time according to Streater's model), a small energy contribution is required for directed transport of Brownian particles to take place.

We remark that, as the number of cycles, k , increases, the energy variation increases too, and, for each cycle $m = 0, \dots, k$, the increment $[E((m+1)t_V) - E(mt_V)]$ is constant. We may interpret this result by saying that the system shows no memory.

We analyzed the variation of entropy. In equation (6.10) we showed that entropy variation is a second-order effect and, as such, it can be neglected within a first-order theory. Thus, we conclude that also in the case of Brownian particles interacting with a changing temperature field in the presence of an adiabatic potential, entropy can be regarded as constant within the adiabatic approximation.

In a forthcoming paper, we would like to abandon the approximation made in equation (4.12) and address this problem by accounting for the non-equilibrium contributions featuring also in the zeroth-order fields, \bar{T}_0 and $\bar{\rho}_0$.

Acknowledgments

The authors gratefully acknowledge Raymond F. Streater for useful discussions and Giandomenico Zingali for his help in revising the manuscript. This work was supported by CNISM—Consorzio Nazionale Interuniversitario di Struttura della Materia (National Inter-university Consortium of Structure of Matter), The Canada Research Chair Programme for Cellular and Molecular Biomechanics and the Canadian Institutes for Health Research (CIHR).

References

- [1] Csahok Z, Family F and Vicsek T 1997 *Phys. Rev. E* **55** 5, 5179-83
- [2] Derenyi I, Bier M and Astumian R D 1999 *Phys. Rev. Lett.* **83** 903-6
- [3] Gillespie D and Eisenberg R 2001 *Phys. Rev. E* **63** 061902
- [4] Van de Vondel J, de Souza Silva C C, Zhu B Y, Morelle M and Moshchalkov V V 2005 *Phys. Rev. Lett.* **94** 057003
- [5] de Souza Silva C C, Van de Vondel J, Zhu B Y, Morelle M and Moshchalkov V V 2006 *Phys. Rev. B* **73** 014507
- [6] de Souza Silva C C, Van de Vondel J, Morelle M and Moshchalkov V V 2006 *Nature* **30** 651-4
- [7] Feynman R P, Leighton R B and Sands M 1963 *The Feynman Lectures on Physics* vol 1 (Reading, MA: Addison-Wesley) chapter 46
- [8] Reimann P 2002 *Phys. Rep.* **361** 57-265
- [9] Linke H, Downton M T and Zuckermann M J 2005 *Chaos* **15** 026111
- [10] Magnasco M O 1993 *Phys. Rev. Lett.* **71** 1477
- [11] Astumian R D and Bier M 1994 *Phys. Rev. Lett.* **72** 1766
- [12] Julicher F and Prost J 1995 *Phys. Rev. Lett.* **75** 2618
- [13] Magnasco M O 1993 *Phys. Rev. Lett.* **71** 10
- [14] Parrondo J M R 1998 *Phys. Rev. E* **57** 6, 7297-300

- [15] Streater R F 1995 *Statistical Dynamics* (London: Imperial College Press)
- [16] Streater R F 1997 *J. Stat. Phys.* **88** 447
- [17] Oshanin G, Klafter J and Urbakh U 2004 *Europhys. Lett.* **68** 26
- [18] Oshanin G, Klafter J and Urbakh U 2005 *J. Phys.: Condensed Matter* **17** S3697
- [19] Reimann P, Bartussek R, Häußler R and Hänggi P 1996 *Phys. Lett. A* **215** 26–31
- [20] Smirnov V I 1964 *A Course of Higher Mathematics: Integral Equation and Partial Differential Equations* (Oxford, London: Pergamon)
- [21] Bensoussan A, Lions J-L and Papanicolau G 1978 *Asymptotic Analysis for Periodic Structures* (Amsterdam: North-Holland)
- [22] Horsthemke W and Lefever R 1984 *Noise-Induced Transitions: Theory and Applications in Physics, Chemistry, and Biology* (Berlin: Springer) Springer Series in Synergetics vol 15